

GENERIC UNIQUENESS OF THE MINIMAL MOULTON CENTRAL CONFIGURATION

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ABSTRACT. We prove that, for generic (open and dense) values of the masses, the Newtonian potential function of the collinear N-body problem has $N!/2$ critical values when restricted to a fixed inertia level. In particular, we prove that for generic values of the masses, there is only one global minimal Moulton configuration.

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1. INTRODUCTION

In the N-body problem there is a family of solutions that conserve the shape in the evolution in time. Among these motions, those with zero angular momentum are called homothetic motions. They have the form

$$x(t) = (r_1(t), \dots, r_N(t)) = \phi(t) x_0$$

where $\phi(t) > 0$ is a solution of a one center problem in the line \mathbb{R}_+ , and x_0 a central configuration. This kind of configurations can be defined in many equivalent ways, say for instance as the critical points of the restrictions of the potential function

$$U(x) = \sum_{i < j} \frac{m_i m_j}{r_{ij}},$$

to the level sets of the moment of inertia

$$I(x) = \sum_{i=1}^N m_i r_i^2.$$

It turns out that these configurations have *center of mass at the origin*. For some authors there is an extended notion of central configuration with respect to its center of mass. These are only translation of the first ones.

In this paper we will be interested in the collinear N-body problem, therefore a configuration $x = (r_1, r_2, \dots, r_N) \in \mathbb{R}^N$ will represent the vector of positions of the bodies, which are supposed to be point particles, each with mass $m_i > 0$, and contained in a straight line. As usual, $r_{ij} = |r_i - r_j|$ will denote the distance between the bodies r_i and r_j .

When the bodies evolve in a space of dimension $k > 1$ not much is known about the geometry of central configurations. Not even known in general if there exist only a finite number – modulo similitude – of central configurations. One of the most recent works on this topic, due to Albouy and Kaloshin [2], shows the generic finiteness in the case of five bodies in the plane, that is, excluding the situation in which the vector of masses $m = (m_1, \dots, m_5)$ belongs to a given subvariety of \mathbb{R}_+^5 .

In contrast, for dimension $k = 1$, the problem is solved. The first step was given by Euler who solved the case of three bodies see [3]. Moulton solved the problem

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for arbitrary number of masses. More precisely, he proved in [7] that if we identify configurations which are homothetic by a positive factor, then there are exactly $N!$ equivalence classes of critical points, each one corresponding to an order $\sigma \in S_N$ of the bodies in the line. As we will explain, they are all nondegenerate local minima. See also the appendix on Moulton's theorem in the paper by Smale [9].

The mass vector $m = (m_1, \dots, m_N) \in \mathbb{R}_+^N$ is a parameter which determines the potential function U and the moment of inertia I . Thus the mass vector also determines the central configurations. Before stating our result, let us recall some well known equivalent definitions of central configuration. Once the mass vector is fixed, we say that a configuration $x_0 \in \mathbb{R}^N$ without collisions (that is, such that $U(x_0) < +\infty$) is a *central configuration* if and only if one of the following equivalent conditions is satisfied:

- (a) x_0 is a critical point of U_0 , the restriction of U to the level set of I which contains x_0 .
- (b) x_0 is a critical point of the homogeneous function (of zero degree)

$$\tilde{U} = U I^{1/2}.$$

- (c) x_0 is a critical point of the function $U + \lambda I$ for some value of $\lambda > 0$.

Note that if x_0 is a central configuration then $r x_0$ is also a central configuration for every $r \neq 0$. Moreover, the notion of *nondegenerate* central configuration refers to the first condition. More precisely, if $I(x_0) = k$ then x_0 is a nondegenerate central configuration when x_0 is a nondegenerate critical point of the restriction of U to the ellipsoid $S_k = \{x \mid I(x) = k\}$.

The main result of the present note is the following theorem and his corollary.

Theorem 1. *There is an open and dense set of mass vectors $A \subset \mathbb{R}_+^N$ such that, if $m \in A$ then the function \tilde{U} has $N!/2$ critical values.*

Corollary 2. *There is an open and dense set of mass vectors for which the collinear N -body problem has only one global minimal configuration.*

Of course, the uniqueness in the statement of the corollary refers to the similarity classes of central configurations, that is to say, once we identify configurations which are homothetic by a non zero factor. Thus there are $N!/2$ different central configurations in this sense, and generically only one of them is minimal.

In contrast, the number of critical values can be less than $N!/2$ for some values of the mass vector. It is clear that if two masses are equal, and $N > 3$, then commutation of the corresponding bodies gives an extra symmetry of the problem which is not induced by an spacial isometry. In that case it is also clear that we must have at least two non similar minimal configurations, and at most $N!/4$ critical values of the potential function restricted to any inertia level. If all the masses are equal, the action of the full symmetric group preserves the set of central configurations, which in turn implies that the restriction of the potential function to any inertia level has only one critical value, that is, the potential takes the same value at every normalized central configuration.

Before beginning the proof of the theorem, let us explain our special interest in minimal central configurations. They appear repeatedly in the recent literature on the general N -body problem. More precisely, the minimality condition is often necessary to apply global variational methods. Indeed, in [6] the second author and Venturelli have proved that if α is a given minimal configuration normalized in the sense that $I(\alpha) = 1$, then for any configuration x_0 there is at least one motion $x(t)$ starting from x_0 which is completely parabolic for $t \rightarrow +\infty$, and whose normalized configuration $x(t) I(x(t))^{-1/2}$ converges to α . More recently, Percino and Sánchez-Morgado [8] built the Busemann functions associated to each minimal

configuration. This last result improves the previous one, because each of these functions is provided with a lamination of completely parabolic motions which are asymptotic to the minimal configuration.

In higher dimensions, as we already said, very little is known about the number of minimal central configurations modulo similitude. However, at the risk of being bold and naive, it seems natural to expect that generically in the masses there should be only one minimal configuration. This is true for instance when the dimension of the Euclidean space in which the bodies move is $k \geq 2$ and the number of bodies N does not exceeds $k + 1$. In this case we have, *for any choice of the masses*, only one minimal configuration in which all the mutual distances r_{ij} are equal. The main result in this work shows that this is also true for arbitrary number of bodies and generic masses in the collinear case.

The proof of theorem 1 is divided in several lemmas which shall be established in the next section. The first two are given for the sake of completeness even if they are well known. More precisely, these two lemmas contain a proof of Moulton's theorem which includes the analytic dependence on the mass vector.

2. PROOF

We begin by recalling a very useful and well known way to normalize central configurations which was proposed by Yoccoz at a conference in Palaiseau ([10]). It is clear that $z \in \Omega$ is a central configuration if and only if there exists $\lambda \in \mathbb{R}$ such that

$$\nabla U(z) + \lambda \nabla I(z) = 0.$$

Since the functions U and I are homogeneous of degree -1 and 2 respectively, we deduce that

$$\begin{aligned} 0 &= \langle \nabla U(z), z \rangle + \lambda \langle \nabla I(z), z \rangle \\ &= -U(z) + 2\lambda I(z) \end{aligned}$$

hence that $\lambda = U(z)/2I(z)$. We also see that z is a central configuration if and only if μz is a central configuration for all $\mu > 0$, and that $\lambda(\mu z) = \mu^{-3}\lambda(z)$. Therefore we conclude that there are two natural ways to normalize the size of a central configuration: fixing the value of the moment of inertia, or fixing the value of λ . The advantage of the second one is that the normalized configuration is a critical point of the function $U + \lambda I$ in the open set Ω rather than a critical point of the restriction of U to some level set of the moment of inertia.

Our first lemma proves the uniqueness and the analytical dependence on the masses, of the normal central configuration once fixed the ordering of the bodies. Let us introduce before some convenient notation.

First, since we will consider varying masses, it will be convenient to use the notation $U_m(x) = U(x, m)$ and $I_m(x) = I(x, m)$ for the values at x of the potential function and the moment of inertia respect to the origin respectively. Note that both functions U and I are real analytic functions in $\Omega \times \mathbb{R}_+^N$.

Finally, as usual, S_N will denote the group of bijections of the set $\{1, \dots, N\}$ into itself. Each element of S_N is therefore identified with an ordering of the N bodies in the oriented straight line. For $\sigma \in S_N$ we define the open set Ω_σ as the set of configurations of N bodies in the oriented line with the ordering prescribed by σ , that is to say,

$$\Omega_\sigma = \{ x = (r_1, \dots, r_n) \mid r_{\sigma(1)} < \dots < r_{\sigma(N)} \}.$$

In other words, $\sigma(i) = j$ means that the mass j occupies the place i from left to right. It is clear that the set $\Omega \subset \mathbb{R}^N$ of configurations without collisions is the disjoint union of the above sets. Thus Ω has $N!$ connected components.

Lemma 1 (Moulton's theorem). *For each $\sigma \in S_N$ there is a real analytic function*

$$x_\sigma : \mathbb{R}_+^N \rightarrow \Omega_\sigma$$

such that $x_\sigma(m)$ is the unique central configuration in Ω_σ for the collinear N -body problem with mass vector m such that $I_m(x_\sigma(m)) = 1$.

Proof. We will prove that for any value of $m \in \mathbb{R}_+^N$ and any $\sigma \in S_N$ the function

$$W_m = U_m + I_m$$

has a unique critical point in Ω_σ . Clearly W_m is a proper function over each convex set Ω_σ . Indeed, for each $K > 0$, $U_m(x) \leq K$ implies that x is in the closed set

$$\{ (r_1, \dots, r_N) \in \mathbb{R}^N \mid K r_{ij} \geq m_i m_j > 0 \text{ for all } 1 \leq i < j \leq N \} \subset \Omega,$$

and $\{x \mid I_m(x) \leq K\}$ is a compact subset of \mathbb{R}^N . On the other hand W_m is strictly convex in Ω . A simple computation shows that

$$\frac{\partial^2 W_m}{\partial r_i^2}(x) = 2m_i + \sum_{k \neq i} 2m_i m_k r_{ik}^{-3} \quad \text{and that} \quad \frac{\partial^2 W_m}{\partial r_i \partial r_j}(x) = -2m_i m_j r_{ij}^{-3}$$

when $i \neq j$. Thus given $x = (r_1, \dots, r_N)$ and $y = (s_1, \dots, s_N)$ we can write

$$\langle y, D^2 W_m(x) y \rangle = 2 \sum_{i < j} m_i m_j r_{ij}^{-3} (s_i - s_j)^2 + 2 I_m(y),$$

which implies that the spectrum of the Hessian matrix is uniformly bounded from below by $2m_0$ where $m_0 = \min\{m_1, \dots, m_N\} > 0$. The same conclusion can be obtained by application of the Gershgorin circle theorem (see [4]). Therefore we deduce that the function W_m has one and only one critical point at each component Ω_σ of Ω . We will call $c_\sigma(m)$ this critical point. We have that $c_\sigma(m)$ is the unique central configuration in Ω_σ such that $\lambda_m(c_\sigma(m)) = 1$.

The map $c_\sigma : \mathbb{R}_+^N \rightarrow \Omega_\sigma$ is real analytic because it is also defined by the real analytic implicit function theorem (see for instance chapter 6 in [5]), applied to the real analytic function

$$F_\sigma : \Omega_\sigma \times \mathbb{R}_+^N \rightarrow \mathbb{R}^N$$

given by

$$F_\sigma(x, m) = \frac{\partial U}{\partial x}(x, m) + \frac{\partial I}{\partial x}(x, m) = \nabla W_m(x).$$

We know that the necessary condition to apply the implicit function theorem is satisfied since

$$\frac{\partial F_\sigma}{\partial x}(x, m) = D^2 W_m(x)$$

is the Hessian matrix of the function W_m and we already know that is positive definite at every point.

In order to finish the proof, we write as a function of m the corresponding central configuration with unitary moment of inertia. Indeed, since $\lambda_m(c_\sigma(m)) = 1$ we have that

$$2 I(c_\sigma(m), m) = U(c_\sigma(m), m),$$

and therefore

$$x_\sigma(m) = \sqrt{2} c_\sigma(m) U(c_\sigma(m), m)^{-1/2}$$

defines a real analytic function which gives, for each value of the mass vector m the unique central configuration in Ω_σ with moment of inertia equal to 1. \square

Now we will prove that the collinear central configurations, also called Moulton configurations, are local minima of $\tilde{U}_m = U_m I_m^{1/2}$. Note that $\tilde{U}_m(x)$ is the value of the potential U_m at the normalized configuration $I_m(x)^{-1/2}x$. Moreover, if we call

$$\mathbb{S}_m = \{x \in \mathbb{R}^N \mid I_m(x) = 1\}$$

then every central configurations in \mathbb{S}_m is a nondegenerate local minimum of the restriction $U_m|_{\mathbb{S}_m}$, and a global minimum on each component $\Omega_\sigma \cap \mathbb{S}_m$. We give the proof of this well known fact for the sake of completeness. We will use the arguments in the proof of the previous lemma.

Lemma 2. *Given $m \in \mathbb{R}_+^N$ and $\sigma \in S_N$ let us write $\Sigma = \Omega_\sigma \cap \mathbb{S}_m$ for the set of normal configurations with order σ . The function $U_m|_\Sigma$ has a unique global minimum which is nondegenerate.*

Proof. We already know that $U_m|_\Sigma$ has a unique critical point, thus we only have to prove that it is a nondegenerate minimum. The critical point is the point x_σ in the previous lemma, so we have

$$x_\sigma = \sqrt{2} c_\sigma U_m(c_\sigma)^{-1/2}$$

where c_σ is the unique critical point of $W_m = U_m + I_m$ in Ω_σ . Now we consider the map $\varphi : \Sigma \rightarrow \Omega_\sigma$ given by

$$\varphi(x) = \left(\frac{U_m(x)}{2} \right)^{1/3} x.$$

Clearly, φ is a smooth embedding which satisfies $\varphi(x_\sigma) = c_\sigma$, as shown in figure 1.

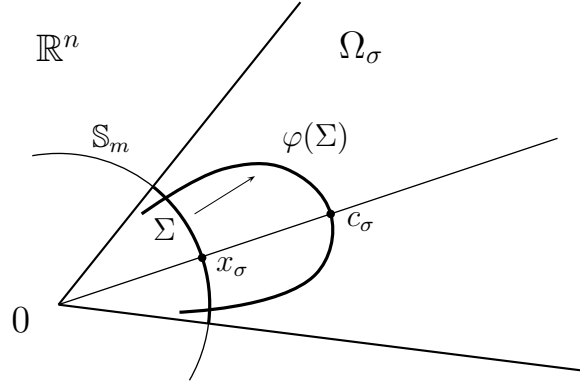


FIGURE 1. The two different normalizations of a central configuration.

Moreover, for any $x \in \Sigma$ we have

$$U_m(\varphi(x)) = 2^{1/3} U_m(x)^{2/3}, \quad I_m(\varphi(x)) = (1/4) U_m(x)^{2/3}$$

hence

$$U_m(x) = a W_m(\varphi(x))^{3/2}$$

for some constant $a > 0$. This proves that x_σ is a nondegenerate minimum of $U_m|_\Sigma$ because W_m has a nondegenerate minimum at $c_\sigma = \varphi(x_\sigma)$ and $W_m(c_\sigma) > 0$. \square

From now on, we will denote $\mathcal{M}_N(\sigma, m)$ the minimal value of the potential function U_m restricted to $\Sigma = \Omega_\sigma \cap \mathbb{S}_m$, the set of normal configurations of N bodies in the oriented line with a given order prescribed by a permutation $\sigma \in S_N$. Thus we have $\mathcal{M}_N(\sigma, m) = U_m(x_\sigma)$.

We will say that $\sigma \in S_{N+k}$ is *compatible* with $\sigma_0 \in S_N$ whenever for every

$$x = (r_1, \dots, r_N, r_{N+1}, \dots, r_{N+k}) \in \Omega_\sigma \subset \mathbb{R}^{N+k}$$

we have

$$y = (r_1, \dots, r_N) \in \Omega_{\sigma_0} \subset \mathbb{R}^N.$$

Of course the condition can be written in terms of σ and σ_0 exclusively. More precisely, taking into account that the value $\sigma(n)$ is the number of the body in the n -th place from, it is easy to see that σ is compatible with σ_0 if and only if the function

$$\sigma|_{\{1, \dots, N\}} \circ \sigma_0^{-1} : \{1, \dots, N\} \rightarrow \{1, \dots, N, N+1, \dots, N+k\}$$

is increasing.

Lemma 3. Assume that $m_0 = (m_1, \dots, m_N) \in \mathbb{R}_+^N$ and $\sigma \in S_N$ are given and that $\tau \in S_{N+K}$ is compatible with σ . If for $\epsilon > 0$ we define the mass vector

$$m(\epsilon) = (m_1, \dots, m_N, \epsilon m_{N+1}, \dots, \epsilon m_{N+K}) \in \mathbb{R}_+^{N+K}$$

then we have

$$\lim_{\epsilon \rightarrow 0} \mathcal{M}_{N+K}(\tau, m(\epsilon)) = \mathcal{M}_N(\sigma, m_0).$$

Proof. Let $(\epsilon_n)_{n>0}$ be a minimizing sequence for $\mathcal{M}_{N+K}(\tau, m(\epsilon))$. This means that $\epsilon_n \rightarrow 0$ and that

$$\liminf_{\epsilon \rightarrow 0} \mathcal{M}_{N+K}(\tau, m(\epsilon)) = \lim_{n \rightarrow \infty} \mathcal{M}_{N+K}(\tau, m(\epsilon_n)).$$

Now, for each $n > 0$, we define $x_n \in \mathbb{R}^{N+K}$ as the unique normalized central configuration of the $N+K$ bodies given by lemma 1, for the mass vector $m(\epsilon_n)$ and the ordering given by τ . Thus, for each $n > 0$ we have

$$I(x_n, m(\epsilon_n)) = 1 \quad \text{and} \quad \mathcal{M}_{N+K}(\tau, m(\epsilon_n)) = U(x_n, m(\epsilon_n)),$$

where the last equality is due to lemma 2. Moreover, if we write

$$x_n = (r_1^n, \dots, r_N^n, r_{N+1}^n, \dots, r_{N+K}^n), \quad \text{and} \quad y_n = (r_1^n, \dots, r_N^n),$$

then the compatibility of τ with σ says that the configuration y_n has the ordering given by the permutation σ . The configuration y_n is not normalized for the vector mass m_0 as it is clear that $I(y_n, m_0) < 1$. However, if we define

$$\alpha_n = m_{N+1} (r_{N+1}^n)^2 + \dots + m_{N+K} (r_{N+K}^n)^2,$$

we can write

$$I(y_n, m_0) = I(x_n, m(\epsilon_n)) - \epsilon_n \alpha_n = 1 - \epsilon_n \alpha_n$$

so the normalization of y_n gives the configuration $z_n = (1 - \epsilon_n \alpha_n)^{-1/2} y_n$, and we have

$$U(z_n, m_0) = (1 - \epsilon_n \alpha_n)^{1/2} U(y_n, m_0).$$

On the other hand, we have that

$$U(x_n, m(\epsilon_n)) = U(y_n, m_0) + \sum_{i=1}^N \sum_{j=N+1}^{N+K} \frac{\epsilon_n m_i m_j}{|r_i^n - r_j^n|} + \sum_{N+1 \leq i < j \leq N+K} \frac{\epsilon_n^2 m_i m_j}{|r_i^n - r_j^n|}.$$

Since $(1 - \epsilon_n \alpha_n)^{1/2} < 1$, we deduce that

$$U(z_n, m_0) < U(y_n, m_0) < U(x_n, m(\epsilon_n)).$$

Thus, given that $\mathcal{M}_N(\sigma, m_0) \leq U(z_n, m_0)$, we conclude that

$$\mathcal{M}_N(\sigma, m_0) < U(x_n, m(\epsilon_n)) = \mathcal{M}_{N+K}(\tau, m(\epsilon_n)).$$

Taking the limit for $n \rightarrow \infty$ we obtain the inequality

$$\mathcal{M}_N(\sigma, m_0) \leq \liminf_{\epsilon \rightarrow 0} \mathcal{M}_{N+K}(\tau, m(\epsilon)).$$

We fix now $\delta > 0$ and we define $z = (r_1, \dots, r_N)$ as the unique normal central configuration for the mass vector m_0 and ordering prescribed by σ . In particular we have $\mathcal{M}_N(\sigma, m_0) = U(z, m_0)$ by lemma 2. We will prove that the inequality

$$\mathcal{M}_{N+K}(\tau, m(\epsilon)) \leq U(z, m_0) + \delta$$

is satisfied whenever $\epsilon > 0$ is small enough. This will finish the proof, since it implies that

$$\limsup_{\epsilon \rightarrow 0} \mathcal{M}_{N+K}(\tau, m(\epsilon)) \leq \mathcal{M}_N(\sigma, m_0).$$

Since τ is compatible with σ , we can add to the configuration $z = (r_1, \dots, r_N)$ the positions of K bodies, in such a way that the ordering of the resulting extended configuration $y = (r_1, \dots, r_N, r_{N+1}, \dots, r_{N+K})$ is given by τ . We shall call r_0 the minimal distance between the positions in the configuration y , that is to say,

$$r_0 = \min \{ |r_i - r_j| \mid 1 \leq i < j \leq N+K \} > 0.$$

We will also consider the moment of inertia of the configuration y with respect to the mass vector $m(\epsilon)$, and we will denote it by I_ϵ . Thus we can write

$$I_\epsilon = I(y, m(\epsilon)) = I(z, m_0) + \epsilon \alpha,$$

where

$$\alpha = m_{N+1} r_{N+1}^2 + \dots + m_{N+K} r_{N+K}^2.$$

Moreover, since z is a normal configuration for m_0 , we can write $I_\epsilon = 1 + \epsilon \alpha$. Thus, normalizing y with respect to the mass vector $m(\epsilon)$ we obtain the configuration

$$x_\epsilon = I_\epsilon^{-1/2} y.$$

We observe now that the homogeneity gives $U(x_\epsilon, m(\epsilon_n)) = I_\epsilon^{1/2} U(y, m(\epsilon))$ and that

$$U(y, m(\epsilon_n)) = U(z, m_0) + \sum_{i=1}^N \sum_{j=N+1}^{N+K} \frac{\epsilon m_i m_j}{|r_i - r_j|} + \sum_{N+1 \leq i < j \leq N+K} \frac{\epsilon^2 m_i m_j}{|r_i - r_j|}.$$

Hence we deduce the upper bound

$$U(x_\epsilon, m(\epsilon)) \leq I_\epsilon^{1/2} \left(U(z, m_0) + N K \frac{\epsilon \mu^2}{r_0} + \frac{K(K-1)}{2} \frac{\epsilon^2 \mu^2}{r_0} \right),$$

where $\mu = \max \{ m_1, \dots, m_{N+K} \}$. Therefore, since the right hand of the previous inequality is a continuous function of ϵ , and $\mathcal{M}_{N+K}(\tau, m(\epsilon)) \leq U(x_\epsilon, m(\epsilon))$, we conclude that there is $\epsilon_0 > 0$ such that

$$\mathcal{M}_{N+K}(\tau, m(\epsilon)) < U(z, m_0) + \delta$$

whenever $\epsilon < \epsilon_0$, as we wanted to prove. \square

Lemma 4. *There is $\mu > 0$ for which*

$$\mathcal{M}_3(id, (1, \mu, 1)) \neq \mathcal{M}_3((2, 3), (1, \mu, 1)) = \mathcal{M}_3(id, (1, 1, \mu)).$$

Proof. Let us first compute $\mathcal{M}_3(id, (1, \mu, 1))$. The symmetry of the mass vector implies that the central configurations in Ω_{id} are also symmetric. This means that the configurations have the form $x_r = (-r, 0, r)$ with $r > 0$. Computing the potential function and the moment of inertia we get

$$I(x_r) = 2r^2 \quad \text{and} \quad U(x_r) = \frac{1}{2r} + \frac{2\mu}{r}.$$

So the normal central configuration for this order of the masses is $(-1/\sqrt{2}, 0, 1/\sqrt{2})$. We deduce that

$$\mathcal{M}_3(id, (1, \mu, 1)) = \frac{\sqrt{2}}{2} + 2\sqrt{2}\mu.$$

The second distribution of masses is not symmetric. However, Euler has showed (see [3], or [1] for a modern reference) that up to a translation and rescale, a central configuration for the mass vector (m_1, m_2, m_3) and order $\sigma = id$ is $(0, 1, 1+s)$, where s is the unique positive root of the polynomial

$$\begin{aligned} p(s) = & -(m_1 + m_2)s^5 - (3m_1 + 2m_2)s^4 - (3m_1 + m_2)s^3 + \\ & +(m_2 + 3m_3)s^2 + (2m_2 + 3m_3)s + (m_2 + m_3). \end{aligned}$$

Since in our case we have $m_1 = m_2 = 1$ and $m_3 = \mu$ the polynomial becomes

$$p(s) = -2s^5 - 5s^4 - 4s^3 + (1 + 3\mu)s^2 + (2 + 3\mu)s + (1 + \mu).$$

We claim that there is $\mu > 0$ for which $(0, 1, 3)$ is a translated central configuration. Therefore $s = 2$ must be a root of this polynomial, which gives rise to the linear equation

$$p(2) = 19\mu - 171 = 0$$

whose solution is $\mu = 9$. We conclude that, the central configurations, for the mass vector $(1, 1, 9)$ and the ordering given by $\sigma = id$, have the form $y_r = (0, r, 3r)$ with $r > 0$. Using the Leibnitz formula for the moment of inertia with respect to the center of mass we avoid to translate the configuration. More precisely, we have

$$\begin{aligned} I_G(y_r) &= \frac{1}{m_1 + m_2 + m_3} (m_1 m_2 r_{12}^2 + m_1 m_3 r_{13}^2 + m_2 m_3 r_{23}^2) \\ &= \frac{1}{11} (r^2 + 9(3r)^2 + 9(2r)^2) = \frac{118}{11} r^2. \end{aligned}$$

In particular, the central configuration with moment of inertia $I_G = 1$ is, up to a translation, the configuration y_r for $r = (11/118)^{1/2}$. Now we can compute the value of the potential function in this configuration, and we get

$$U(y_r) = \frac{1}{r} + \frac{9}{2r} + \frac{9}{3r} = \frac{51}{6} \left(\frac{118}{11} \right)^{1/2}.$$

Therefore the lemma is proved, since for $\mu = 9$ we have computed

$$\mathcal{M}_3(id, (1, 9, 1)) = \frac{37}{\sqrt{2}}$$

and

$$\mathcal{M}_3((2, 3), (1, 9, 1)) = \mathcal{M}_3(id, (1, 1, 9)) = \frac{51}{6} \left(\frac{118}{11} \right)^{1/2}.$$

□

The last lemma we will need in the proof of the theorem is purely combinatorial and characterizes the fact that two permutations are not equal nor symmetric. Let us introduce first simplifying notations. If $\sigma \in S_N$ is a given permutation, then we will write $\bar{\sigma}$ to denote the permutation corresponding to the inverse order. More precisely, $\bar{\sigma}$ is defined by $\bar{\sigma}(k) = \sigma(N + 1 - k)$. Moreover, given $\sigma \in S_N$ and numbers $i, j, k \in \{1, \dots, N\}$, we will say that $\sigma(i)$ is between $\sigma(j)$ and $\sigma(k)$ if either $\sigma(j) < \sigma(i) < \sigma(k)$ or $\sigma(k) < \sigma(i) < \sigma(j)$.

Lemma 5. *If σ and τ are two given permutations then we have the following alternative: either $\sigma = \tau$, $\sigma = \bar{\tau}$, or there are three numbers $i, j, k \in \{1, \dots, N\}$ such that $\sigma(i)$ is between $\sigma(j)$ and $\sigma(k)$, but $\tau(i)$ is not between $\tau(j)$ and $\tau(k)$.*

Proof. Clearly, each one of the first two possibilities in the triple alternative excludes the others. Thus it suffices to show that if the third possibility is not satisfied then one of the two first must be true.

It is not difficult to see that if the third possibility is not satisfied then $\sigma \circ \tau^{-1}$ is a monotone bijection. On the other hand, the only permutations on the n numbers $\{1, \dots, n\}$ which are monotone are the identity and the inversion. Thus, we must have $\sigma = \tau$ or $\sigma = \bar{\tau}$. \square

Proof of theorem 1. Recall that for each $\sigma \in S_N$, we denote $x_\sigma(m)$ the unique central configuration with ordering given by σ and normalized in the sense that $I(x_\sigma(m), m) = 1$. Therefore, the set of critical values of the function \tilde{U} is exactly

$$V_c(m) = \{ U(x_\sigma(m), m) \mid \sigma \in S_N \}.$$

Thus we know that the number of critical values is a lower semicontinuous function of $m \in \mathbb{R}_+^N$, so in particular it is a continuous function over the set of maxima

$$A = \{ m \in \mathbb{R}_+^N \text{ such that } |V_c(m)| = N!/2 \}$$

from which we conclude that this set is open.

In what follows we prove that A is dense in \mathbb{R}_+^N . Let us define, for each pair of permutations $\sigma, \tau \in S_N$, the set

$$M_{\sigma, \tau} = \{ m \in \mathbb{R}_+^N \mid U(x_\sigma(m), m) \neq U(x_\tau(m), m) \}.$$

As a consequence of the analyticity property proved in lemma 1 we know that each one of these sets is either open and dense, or empty. We will prove that the empty case happens only if $\sigma = \tau$ or $\sigma = \bar{\tau}$. The proof of this claim finish the proof, since

$$A = \bigcap_{(\sigma, \tau) \in \mathcal{F}} M_{\sigma, \tau}$$

where \mathcal{F} is the set of pairs (σ, τ) of non symmetric permutations, i.e. such that $\sigma \neq \tau$ and $\sigma \neq \bar{\tau}$. In order to prove the claim, we assume by contradiction that σ and τ are non symmetric permutations and that the set $M_{\sigma, \tau}$ is however empty. Thus we have $U(x_\tau(m), m) = U(x_\sigma(m), m)$ for all $m \in \mathbb{R}_+^N$.

On the other hand, since $\sigma \neq \tau$ and $\sigma \neq \bar{\tau}$ by lemma 5 (applied to the inverse permutations σ^{-1} and τ^{-1}) we can assume without loss of generality that there are numbers $1 \leq i < j < k \leq N$ such that

$$\sigma^{-1}(i) < \sigma^{-1}(j) < \sigma^{-1}(k),$$

and

$$\tau^{-1}(i) < \tau^{-1}(k) < \tau^{-1}(j).$$

We can also assume, renumbering the bodies if necessary, $i = 1$, $j = 2$ and $k = 3$. Now consider for small $\epsilon > 0$ the mass vector $m_\epsilon = (1, \mu, 1, \epsilon, \dots, \epsilon)$ given by where μ is the value of the mass given by lemma 4. By lemma 2 we have

$$\mathcal{M}_N(\sigma, m_\epsilon) = U(x_\sigma(m_\epsilon), m_\epsilon) = U(x_\tau(m_\epsilon), m_\epsilon) = \mathcal{M}_N(\tau, m_\epsilon)$$

for all $\epsilon > 0$. Moreover applying lemma 3 with $N = 3$, $\sigma_0 = id$ and $\tau_0 = (2, 3)$ we have

$$\lim_{\epsilon \rightarrow 0} \mathcal{M}_N(\sigma, m_\epsilon) = \mathcal{M}_3(id, (1, \mu, 1))$$

and

$$\lim_{\epsilon \rightarrow 0} \mathcal{M}_N(\tau, m_\epsilon) = \mathcal{M}_3((2, 3), (1, \mu, 1)).$$

This is impossible since it contradicts lemma 4. \square

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